ON SPECIAL PROPERTIES OF A PENDULUM SYSTEM

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A study is made of the plane disturbed motion of a system consisting of a rigid straight rod with a certain number of stable pendulums of equal length hung on the longitudinal axis of the rod. The rod is subjected to a force which is always directed along the rod's longitudinal axis. Such a system of pendulums is nonconservative and becomes unstable for certain relationships between its parameters. In this work there are given these relationships; it is shown that they can be realized under laboratory conditions, and a cause is given that will produce instability of the system.

Let us denote by m the mass of the rod with the pendulums, by I the moment of inertia of the rod with the fastened pendulums relative to the transverse axis passing through the center of gravity of the system, by P the force acting on the rod, by m_n the mass of the nth pendulum, by L_n the distance from the center of gravity of the undisturbed system to this mass m_n (the distance is considered positive if the mass m_n lies above the center of gravity), by l the length of the pendulums, and by k their number.

The vertical motion of the rod with constant acceleration

$$\ddot{y}_c = \frac{P}{m} - g$$

will be considered as the nondisturbed motion. We shall treat x_c, ϑ, λ_n. as small perturbations of the system (Fig. 1). It is assumed that the perturbations x_c, ϑ, λ_n have at most a second order effect on the acceleration $\ddot{y}_c$. The point C is kept fixed on the longitudinal axis of the rod, and it coincides with the center of gravity of the system when $\lambda_n = 0$.

If we denote the distance from the point C to any point of the rod by y_1 ($y_1 > 0$, when it is measured in the direction of the vector of the force P) then the position of any point of the rod in the absolute system of coordinates xoy will be

$$x = x_c + y_1\vartheta, \qquad y = y_c + y_1\left(1 - \frac{\vartheta^2}{2}\right) \qquad (-l_1 < y_1 < +l_2)$$

The coordinates of the pendulum are

$$x_n = x_c + L_n\vartheta + \lambda_n, \qquad y_n = y_c + L_n\left(1 - \frac{\vartheta^2}{2}\right) + \frac{\lambda_n^2}{2l} - \lambda_n\vartheta$$

The expressions of the coordinates x, x_n and y, y_n are written down with an accuracy to the first and second order, respectively.

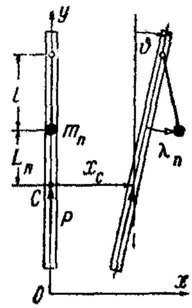

Fig. 1.

The kinetic and potential energies of the system can be expressed as the sum of the energy of the rod and of the pendulums:

$$T = \frac{1}{2}\left[\int_{-l_1}^{+l_2} m(y_1)(\dot{x}^2 + \dot{y}^2)\,dy_1 + \sum_{n=1}^{n=k} m_n(\dot{x}_n^2 + \dot{y}_n^2)\right]$$

$$U = g\int_{-l_1}^{+l_2} m(y_1)\,y\,dy_1 + g\sum_{n=1}^{n=k} m_n y_n$$

The linearized equations of the perturbed motion will have the form

$$m\ddot{x}_c + \sum_{n=1}^{n=k} m_n\ddot{\lambda}_n = P\vartheta, \qquad I\ddot{\vartheta} + \sum_{n=1}^{n=k} m_n L_n\ddot{\lambda}_n - \frac{P}{m}\sum_{n=1}^{n=k} m_n\lambda_n = 0 \tag{1}$$

$$\ddot{\lambda}_n + \frac{P}{lm}\lambda_n + L_n\ddot{\vartheta} - \frac{P}{m}\vartheta + \ddot{x}_c = 0 \qquad (n = 1, \ldots, k)$$

Let us set

$$\ddot{x}_c = Me^{rt}, \qquad \vartheta = Ne^{rt}, \qquad \lambda_n = Q_n e^{rt}, \qquad r = \omega_0 p \qquad \left(\omega_0 = \sqrt{\frac{P}{lm}}\right)$$

Then the characteristic equation of the system (1) can be represented in the form

$$p^2(a_0p^4 + a_1p^2 + a_2) = 0 \tag{2}$$

Here the numbers a_0, a_1 and a_2 are real numbers, μ_1, μ_2, μ_3 and ρ are dimensionless parameters of the system:

$$a_0 = (1 - \mu_1)(1 - \mu_2) - \mu_3^2, \qquad a_1 = 2 - \mu_1 - \mu_2 + \frac{\mu_3}{\rho}, \qquad a_2 = 1 + \frac{\mu_3}{\rho}$$

$$\mu_1 = \frac{1}{m}\sum_{n=1}^{n=k} m_n, \qquad \mu_2 = \frac{1}{I}\sum_{n=1}^{n=k} m_n L_n^2, \qquad \mu_3 = \frac{1}{m\rho_0}\sum_{n=1}^{n=k} m_n L_n$$

$$\rho = \rho_0 / l, \qquad \rho_0 = \sqrt{I/m}$$

In accordance with our notation, the bounds on the parameters will be

$$0 < \mu_1 < 1, \qquad 0 < \mu_2 < 1, \qquad -1 < \mu_3 < 1, \qquad \mu_3^2 \leqslant \mu_1\mu_2, \qquad 0 < \rho < \infty$$

When the pendulums are all of the same length then the highest possible degree of Equation (2) is six.

The value of the roots $p^2 = 0$ in Equation (2) will introduce no singularities. It corresponds to the motion of the system when all $\lambda_n \equiv 0$, and $M = NP/m$. Let us, therefore, consider the second factor:

$$a_0 p^4 + a_1 p^2 + a_2 = 0 \tag{3}$$

For large values of ρ the discriminant of Equation (3)

$$D = 4a_0 a_2 - a_1^2 = bc \qquad (b = 2\sqrt{a_0 a_2} - a_1, \ c = 2\sqrt{a_0 a_2} + a_1)$$

is negative, and the system has two frequencies for its characteristic oscillations:

$$\omega_1^2 = \frac{a_1 + \sqrt{-D}}{2a_0}, \qquad \omega_2^2 = \frac{a_1 - \sqrt{-D}}{2a_0}$$

If, furthermore,

$$\left(\frac{\mu_3}{\mu_1 - \mu_2}\right)^2 \ll 1$$

then the first frequency ω_1, and the second one ω_2 will differ very little from the partial frequencies

$$\omega_1^* = \frac{1}{\sqrt{1 - \mu_1}}, \qquad \omega_2^* = \frac{1}{\sqrt{1 - \mu_2}}$$

which the system has when $\vartheta \equiv 0$, or $\ddot{x}_c \equiv 0$. When $\mu_3 = 0$

$$\omega_1 = \omega_1^*, \qquad \omega_2 = \omega_2^*$$

With a decrease of ρ, when $\mu_3 > 0$, the system preserves its earlier properties. An increase in the parameter μ_3/ρ causes only an increase in the difference between the frequencies ω_1 and ω_2.

For small values of ρ and $\mu_3 < 0$, the discriminant D of Equation (3) can become positive, and the system will possess essentially different properties. The roots $p_{1,2}^2$ of the biquadratic equation (3) will be complex conjugates with negative real parts. In the transition to the final p we obtain

$$p_{1,2} = -\varepsilon \pm j\omega, \qquad p_{3,4} = +\varepsilon \pm j\omega \qquad \left(\varepsilon = \sqrt{\frac{b}{2a_0}}, \ \omega = \sqrt{\frac{c}{2a_0}}\right) \tag{4}$$

We note that two of the complex conjugate roots of the solution (4) will have positive real parts. The characteristic oscillations with frequency ω will be increasing.

Since $\mu_3{}^2 < \mu_1\mu_2$, it can happen that $D > 0$ when $a_2 > 0$. The latter indicates that the solution (4) can exist if the parameter μ_3/ρ lies within the bounds

$$-1 < \frac{\mu_3}{\rho} < 0$$

The condition $b > 0$ can be fulfilled if

$$\frac{\mu_3}{\rho} \approx -(\mu_1 + \mu_2), \qquad \mu_1\mu_2 > \mu_3^2$$

The increment of the oscillations will increase with an increase in $\mu_1\mu_2 - \mu_3{}^2$ and $\mu_1 + \mu_2$. For the purpose of clarifying the mechanism of the increase of the oscillations, we shall express the equation of the system on the basis of (3) in the form

$$\ddot{z} + \frac{a_1}{a_0} z = -\frac{a_2}{a_0} \int_0^t \int_0^t z \, dt^2 \tag{5}$$

This equation can be considered as an equation of forced oscillations of a system with one degree of freedom. The imposing force is hereby proportional to the double integral of the displacement. We shall agree to refer to this force as an external force.

In case $D < 0$, the oscillation will be harmonic, and we will obtain, on the basis of (5), the equation

$$\frac{a_1}{a_0} = \omega^2 + \frac{a_2}{a_0}\frac{1}{\omega^2} \tag{6}$$

The left-hand side of the equation characterizes the spring force, while the right-hand side stands for the force of inertia of the system, and for the external force, which act on the phase when $a_2 > 0$.

To different relationships between the parameters of the system with $\mu_3 > 0$, there correspond different frequencies ω_1 and ω_2 but the relationships between the spring force and the external force must satisfy Equation (6). The oscillations remain stationary because the external force does not increase the energy of the system. The inertia force is decreased by the amount of the external force.

The oscillations with the frequencies $\omega < 1$ and $\omega > 1$ can be represented schematically as the oscillations of a pendulum relative to points lying above or below the point of support (Fig. 2).

If $\mu_3 = 0$, then $\omega_1 = \omega_1^*$ and $\omega_2 = \omega_2^*$. If μ_3/ρ is decreased, then the spring force will decrease more rapidly than the coefficient a_2/a_0, and the low and high frequencies will approach each other. When $D = 0$, the force of inertia is equal to the external force. A further decrease of μ_3/ρ will lead to the situation when the right-hand side of Equation (6) becomes larger than the left-hand side, and the elastic force of the spring cannot balance the inertia and external forces when the amplitude is unchanged. The work of the external force will be absorbed by the increase of the oscillations; the oscillations will grow. The pendulums of the system are those mechanisms by means of which the work done by the external force is used up by the system.

If there is only one pendulum connected to the rod, then the characteristic equation will have the form

$$p^2(1-\mu_1-\mu_2)+1+\frac{\mu_3}{\rho}=0$$

and when $\mu_3/\rho > -1$, the system has one frequency of characteristic oscillations

$$\omega^2=\frac{1+\mu_3/\rho}{1-\mu_1-\mu_2}$$

When $\mu_3/\rho < -1$, the motion of the system is unstable, and the instability is not periodic.

If all the pendulums have slight friction which is proportional to $\varepsilon_n\lambda_n$, the the characteristic equation (2) will contain terms of odd powers of p, and it can then be written in the form

$$p^2(p^2+\varepsilon_1 p+\omega_1^2)(p^2+\varepsilon_2 p+\omega_2^2)=0$$

Computations show that if $D < 0$, then $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$. When $D > 0$, either ε_1 or ε_2 can be negative if $\varepsilon > \varepsilon_n$.

It is easy to realize this type of pendulum system in the laboratory. For this the rod with the pendulums is suspended at the point C on the transverse axis of rotation which can move without friction along horizontal lines in the xoy plane. The rod is connected with a servodrive that can produce a horizontal force P_x proportional to the angle ϑ of the rod's rotation. The force P_x is applied to the axis of rotation of the rod, and acts in the plane xoy, while $P_x = mg\,\vartheta$.

If the position of the laboratory pendulums is determined by the coordinates $x_c(t)$, $\vartheta(t)$, $\lambda_n(t)$, and $y_c = \text{const}$, then the linearized equations of the perturbed motion of the model will be the same as Equations (1) if one sets $P = mg$ in the latter. The dimensionless parameters μ_1, μ_2, μ_3 and ρ can be called coefficients of similarity; $\omega_0 = \sqrt{(g/l)}$ is the scale of the frequency.

For the generation of growing oscillations, an influx of energy is required for the system. The source of such energy in the laboratory model is the servodrive; in the original system, it is the force P. These forces do work to displace x_c when the angle ϑ appears:

$$A = P \int_0^t \vartheta \dot{x}_c \, dt$$

Setting

$$\dot{x}_c = \frac{1}{\omega_0 p} M e^{\omega_0 p t}, \qquad M = N l \omega_0^2 \eta(p)$$

where $\eta(p)$ is a dimensionless coefficient of the amplitude distribution, we obtain

$$A = Pl \int_0^t \sum_{\alpha=1}^{\alpha=4} N_\alpha e^{p_\alpha t} \sum_{\alpha=1}^{\alpha=4} N_\alpha \eta_\alpha (p_\alpha) e^{p_\alpha t} \, dt$$

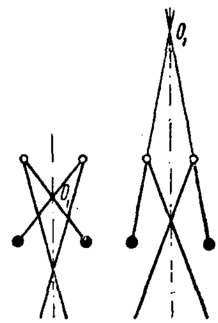

Fig. 2.

The work A will increase indefinitely only then when among the roots p_α of the characteristic equation of the system there exists at least one positive, or two complex conjugate roots with positive real parts. In these cases the energy, which is supplied to the system by the external force, will be used by the system, and the amplitudes of the coordinates x_c, ϑ, and λ_n will increase.

Translated by H.P.T.